Gayatri Vidya Parishad College Of Engineering for Women
(Autonomous)
(Affiliated to Andhra University, Visakhapatnam)

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LINEAR ALGEBRA AND VECTOR CALCULUS
(Common to All Branches)

KEY

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1. a. Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ by reducing it to echelon form.

Sol: $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$R_2 : R_2 - 2R_1$, $R_3 : R_3 - 3R_1$, $R_4 : R_4 - 6R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_3 : R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & +3 & -2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 : R_4 + R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in echelon form.

Rank of $A = \rho(A)$

= No. of non-zero rows

= 3

1. b) Find the values of λ and μ so that the equations $2x+3y+5z=9$, $7x+3y-2z=8$, $2x+3y+\lambda z=\mu$ has infinite number of solutions.

Sol: Augmented matrix $[A|B] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$

$R_2 : 2R_2 - 7R_1$, $R_3 : R_3 - R_1$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -39 & 47 \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix}$$

A non-homogeneous system $AX = B$ is consistent and will have infinite number of solutions if $\rho(A) = \rho(A|B) = r < n$ = number of unknowns.

Here to have infinite no. of solutions, we must have $\lambda = 5$ & $\mu = 9$.

Then $\rho(A) = \rho(A|B) = 2 < 3$ = no. of unknowns.

Thus the required values of λ and μ are

$\lambda = 5$ & $\mu = 9$.

2 a) Apply Gauss elimination method to solve the equations

$x + 4y - z = -5, \quad x + y - 6z = 12, \quad 3x - y - z = 4$

Sol: $[A|B] = \begin{bmatrix} 1 & 4 & -1 & -5 \\ 1 & 1 & -6 & 12 \\ 3 & -1 & -1 & 4 \end{bmatrix}$

$R_2 : R_2 - R_1, \quad R_3 : R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 4 & -1 & -5 \\ 0 & -3 & -5 & 17 \\ 0 & -13 & 2 & 19 \end{bmatrix}$$

$R_3 : 3R_3 - 13R_2$

$$\sim \begin{bmatrix} 1 & 4 & -1 & -5 \\ 0 & -3 & -5 & 17 \\ 0 & 0 & 71 & -164 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x + 4y - z &= -5 \\ -3y - 5z &= 17 \\ 71z &= -164 \end{aligned}$$

By back substitution, $z = -2.3098$,

$y = -1.817$

$x = -0.0418$

2b) Apply factorization method to solve the equations

$3x + 2y + 7z = 4, \quad 2x + 3y + z = 5, \quad 3x + 4y + z = 7.$

Sol: $A = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$

Let $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = A$$

On Comparison we get

$u_{11} = 3$, $u_{12} = 2$, $u_{13} = 7$, $l_{21} = 2/3$, $u_{22} = 5/3$,
$u_{23} = -11/3$, $l_{31} = 1$, $l_{32} = 6/5$, $u_{33} = -8/5$.

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix}$$

The given system now can be written as $LUX = B$

Letting $UX = V$, $AX = B$ becomes $LV = B$

Consider $LV = B \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$

By forward substitution, $v_1 = 4$, $v_2 = \frac{7}{3}$, $v_3 = 1/5$.

Now Consider $UX = V$ then

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$$

By backward substitution, $z = -\frac{1}{8}$, $y = \frac{9}{8}$, $x = \frac{7}{8}$

3 a) Verify Cayley - Hamilton Theorem for the matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and find its inverse.

Sol: Cayley Hamilton Theorem states that "Every Square matrix satisfies its own Characteristic Equation".

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 7\lambda - 2 = 0$$

$$A^2 = \begin{bmatrix} 16 & 21 \\ 28 & 37 \end{bmatrix}$$

Now $A^2 - 7A - 2I = \begin{bmatrix} 16 & 21 \\ 28 & 37 \end{bmatrix} - 7\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Thus A satisfies its own characteristic equation & hence Cayley - Hamilton Theorem is verified.

Consider $A^2 - 7A - 2I = 0$

Premultiplying with A^{-1}, $A - 7I - 2A^{-1} = 0$

$$\Rightarrow A^{-1} = \frac{1}{2}[A - 7I]$$

$$= \frac{1}{2}\begin{bmatrix} -5 & 3 \\ 4 & -2 \end{bmatrix}$$

3b). Find the Singular value decomposition of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Sol: $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Characteristic equation of $A^T A$ is $|A^T A - \lambda I| = 0$.

i.e., $\lambda^2 - 4\lambda + 3 = 0$

$\Rightarrow \lambda = 1, 3$.

$\therefore$ Singular values of $A = \sqrt{3}, 1$

Let $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$

Eigen vector Corresponding to $\lambda = 3$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and eigen vector Corresponding to $\lambda = 1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

These two eigen vectors of $A^T A$ are pairwise orthogonal.

The orthonormal vectors of $A^T A$ are $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $v_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$$\therefore V_{2\times 2} = [v_1 \ v_2] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$\& \ \Sigma_{3\times 2} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Now $U = [u_1 \ u_2 \ u_3]$ where $u_i = \frac{1}{\sigma_i} A v_i$ for $i = 1, 2, 3$

$$u_1 = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \ \& \ u_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Let $u_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be orthogonal to u_1 & u_2

Then $\frac{2a}{\sqrt{6}} + \frac{b}{\sqrt{6}} + \frac{c}{\sqrt{6}} = 0$ and $\frac{b}{\sqrt{2}} - \frac{c}{\sqrt{2}} = 0$

$\Rightarrow 2a + b + c = 0$ & $b - c = 0$

$\Rightarrow b = c$ and $a = -b$

$\therefore u_3 = b \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Normalized vector $u_3 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$$\therefore U = [u_1 \; u_2 \; u_3] = \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

Thus the singular value decomposition of A is given by

$$A = U\Sigma V^T = \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

4. Find the eigen values and eigen vectors of the matrix A and A^{-1} where $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Sol: The characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$$

$\Rightarrow \lambda = -2, 3, 6$ are the eigen vectors of A.

Eigen vectors Corresponding to $\lambda = -2, 3, 6$ are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ respectively.

Result: If λ is an eigen value of A then $1/\lambda$ is an eigen value of $\bar{A}^1$.

Proof: Let λ be an eigen value of A.

Then $AX = \lambda X$.

$$\Rightarrow \bar{A}^1(AX) = \bar{A}^1(\lambda X)$$

$$\Rightarrow IX = \lambda(\bar{A}^1 X)$$

$$\Rightarrow \bar{A}^1 X = \frac{1}{\lambda} X$$

This shows that $\frac{1}{\lambda}$ is an eigen value of $\bar{A}^1$ with X as the Corresponding eigen vector.

Thus the eigen values of the matrix $\bar{A}^1$ are $-\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ and the eigen vectors are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ respectively

5. Reduce the quadratic form $3x^2+5y^2+3z^2-2yz+2zx-2xy$ into the canonical form by an orthogonal reduction and find its nature.

Sol: Matrix of the quadratic form = $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Characteristic equation of A is $|A-\lambda I| = 0$

$\Rightarrow \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

$\Rightarrow \lambda = 2, 3, 6.$

Eigen vector corresponding to $\lambda = 2$:-

Consider $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$R_2 : R_2 + R_1$, $R_3 : R_3 - R_1$

$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow x_1 - x_2 + x_3 = 0$ & $2x_2 = 0$

$\Rightarrow x_2 = 0$ & $x_1 = -x_3 = \alpha$ (say)

$\therefore X_1 = \alpha \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Similarly the eigen vectors corresponding to $\lambda = 3$ & $\lambda = 6$ are $X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ respectively.

The eigen vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ are pairwise orthogonal.

$\therefore$ The normalized modal matrix $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$

is orthogonal & hence $P^T = P^{-1}$.

$$P^{-1}AP = P^TAP = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = D \text{ (Spectral matrix)}$$

Consider the orthogonal transformation $X = PY$

Then $X^TAX = (PY)^T A (PY)$

$$= (Y^TP^T)A(PY)$$

$$= Y^T(P^TAP)Y$$

$$= Y^TDY$$

$$= 2y_1^2 + 3y_2^2 + 6y_3^2$$

Nature of the quadratic form is positive definite.

6. Reduce the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ to the diagonal form and find A^4.

Sol: The characteristic equation of A is $|A - \lambda I| = 0$

i.e., $(2-\lambda)(\lambda^2 - 4\lambda + 3) = 0$

$\Rightarrow (2-\lambda)(\lambda-1)(\lambda-3) = 0$

$\Rightarrow \lambda = 1, 2, 3$ are the eigen values of A.

Eigen vector Corresponding to $\lambda = 1$:-

Consider $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$R_3 : R_3 - R_1$

$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow x_1 + x_3 = 0$ & $x_2 = 0$

$\Rightarrow x_2 = 0$ and $x_1 = -x_3 = \alpha$ (say)

$\therefore X_1 = \alpha \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Similarly the eigen vectors Corresponding to $\lambda = 2$ and $\lambda = 3$ are $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ respectively.

The eigen vectors are pairwise orthogonal and the matrix $P = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$ is orthogonal.

i.e., $P^T = P^{-1}$

Also $\bar{P}^{1} A P = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\bar{P}^{1} A P = D \Rightarrow A^n = P D^n \bar{P}^{1}$

$\therefore A^4 = P D^4 \bar{P}^{1}$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 81 \\ 0 & 16 & 0 \\ -1 & 0 & 81 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 82 & 0 & 80 \\ 0 & 32 & 0 \\ 80 & 0 & 82 \end{bmatrix}$$

$$= \begin{bmatrix} 41 & 0 & 40 \\ 0 & 16 & 0 \\ 40 & 0 & 41 \end{bmatrix}$$

7. a) Find the directional derivative of $f(x,y,z) = x^2-y^2+2z^2$ at $P(1,2,3)$ in the direction of the vector PQ, where Q is the point $(5,0,4)$.

Sol: Let $\phi : x^2-y^2+2z^2$

$$\nabla\phi = 2x\bar{i} - 2y\bar{j} + 4z\bar{k}$$

$$\nabla\phi\big|_{(1,2,3)} = 2\bar{i} - 4\bar{j} + 12\bar{k}$$

$$\overline{PQ} = \overline{OQ} - \overline{OP} = (5\bar{i}+4\bar{k}) - (\bar{i}+2\bar{j}+3\bar{k})$$

$$= 4\bar{i} - 2\bar{j} + \bar{k} = \bar{a} \text{ (say)}$$

The directional derivative of ϕ at P in the direction of the vector $\bar{a}$ is given by $\frac{d\phi}{dr} = \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$\therefore$ Required directional derivative

$$= (2\bar{i} - 4\bar{j} + 12\bar{k}) \cdot \frac{(4\bar{i} - 2\bar{j} + \bar{k})}{\sqrt{21}}$$

$$= \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

7b) Prove that $\nabla r^n = n r^{n-2}\bar{r}$, where $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$.

Sol:- $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ and $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$$\Rightarrow r^2 = x^2+y^2+z^2$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r} \;\&\; \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now $\nabla r^n = \sum \bar{i} \frac{\partial}{\partial x} r^n$

$$= \sum \bar{i}\, n r^{n-1} \frac{\partial r}{\partial x}$$

$$= \sum \bar{i}\, n r^{n-1} \left(\frac{x}{r}\right)$$

$$= \sum n r^{n-2} x \bar{i}$$

$$= n r^{n-2} (x\bar{i} + y\bar{j} + z\bar{k})$$

$$= n r^{n-2} \bar{r}$$

8 a) Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z = x^2+y^2-3$ at the point $(2,-1,2)$

Sol: Let $\phi : x^2+y^2+z^2-9$ & $\psi = x^2+y^2-z-3$

$\nabla\phi = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$ & $\nabla\psi = 2x\bar{i} + 2y\bar{j} - \bar{k}$

At the point $(2,-1,2)$, the normals are given by

$\bar{n}_1 = 4\bar{i} - 2\bar{j} + 4\bar{k}$ and $\bar{n}_2 = 4\bar{i} - 2\bar{j} - \bar{k}$

If θ is the angle between the normals then

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1|\,|\bar{n}_2|} = \frac{(4\bar{i} - 2\bar{j} + 4\bar{k})\cdot(4\bar{i} - 2\bar{j} - \bar{k})}{6\sqrt{21}}$$

$$= \frac{8}{3\sqrt{21}}$$

8 b) Show that $\nabla\cdot(f\bar{A}) = f(\nabla\cdot\bar{A}) + \bar{A}\cdot(\nabla f)$ where f is a scalar function and $\bar{A}$ is vector function.

Sol: $\nabla\cdot(f\bar{A}) = \sum \bar{i}\cdot\frac{\partial}{\partial x}(f\bar{A})$

$$= \sum \bar{i}\cdot\left[\frac{\partial f}{\partial x}\bar{A} + f\frac{\partial \bar{A}}{\partial x}\right]$$

$$= \sum\left[\bar{i}\cdot\frac{\partial f}{\partial x}\bar{A} + \bar{i}\cdot f\frac{\partial \bar{A}}{\partial x}\right]$$

$$= \sum\left[\left(\bar{i}\frac{\partial f}{\partial x}\cdot\bar{A}\right) + f\left(\bar{i}\cdot\frac{\partial \bar{A}}{\partial x}\right)\right]$$

$$= \nabla f\cdot\bar{A} + f(\nabla\cdot\bar{A})$$

9 a) Find the work done in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\,k$ along the straight line from $(0,0,0)$ to $(2,1,3)$.

Sol: Work done $= \int_C \bar{F}\cdot d\bar{r}$

$$= \int_C 3x^2dx + (2xz-y)dy + z\,dz$$

$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t$

$\Rightarrow x = 2t,\ y = t,\ z = 3t$

$\Rightarrow t$ varies from 0 to 1

$$= \int_{t=0}^{1} 3(2t^2)2dt + (2(2t)(3t) - t)dt + 3t\cdot 3dt$$

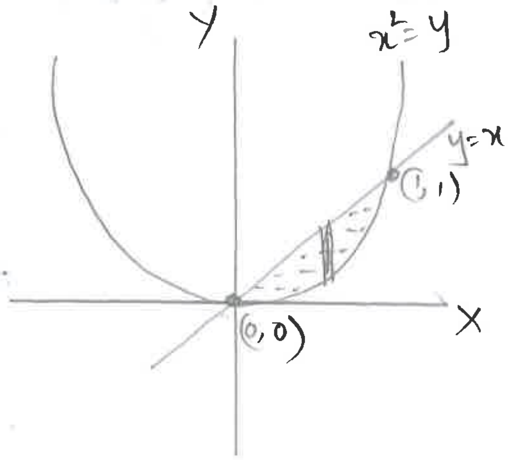
$$= \int_{t=0}^{1}\left[24t^2 + (12t^2 - t) + 9t\right]dt$$

$$= \left[24\frac{t^3}{3} + 12\frac{t^3}{3} - \frac{t^2}{2} + 9\frac{t^2}{2}\right]_0^1$$

$$= 8 + 4 - \frac{1}{2} + 9/2 = 16$$

9b) Using Green's theorem, evaluate $\oint_C (xy+y^2)dx + x^2 dy$ where C is bounded by $y=x$ and $y=x^2$.

<u>Sol</u>: Green's theorem states that



$$\oint_C \phi\,dx + \psi\,dy = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$$

Here $\phi = xy + y^2$ and $\psi = x^2$

$$\frac{\partial \phi}{\partial y} = x + 2y \quad \& \quad \frac{\partial \psi}{\partial x} = 2x$$

$$\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} = x - 2y$$

$$\therefore \int_C (xy+y^2)\,dx + x^2 dy = \iint_R (x-2y)\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x-2y)\,dy\,dx$$

$$= \int_{x=0}^{1} (xy - y^2)_{x^2}^{x}\,dx$$

$$= \int_{x=0}^{1} (x^4 - x^3)\,dx$$

$$= \left(\frac{x^5}{5} - \frac{x^4}{4}\right)_0^1$$

$$= \frac{1}{5} - \frac{1}{4}$$

$$= -\frac{1}{20}$$

10. a) Apply Gauss Divergence Theorem to find $\iint_S \bar{F}\cdot\bar{N}\,ds$ where $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$ taken over the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$.

Sol: Gauss Divergence theorem states that

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv$$

Given $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$

$$\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = 4z - y$$

$$\therefore \iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V (4z-y)\,dx\,dy\,dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4z-y)\,dx\,dy\,dz$$

$$= 4(x)_0^1\,(y)_0^1\,\left(\frac{z^2}{2}\right)_0^1 - (x)_0^1\,(y^2/2)_0^1\,(z)_0^1$$

$$= 2 - \frac{1}{2}$$

$$= 3/2$$

10 b) Using Stoke's theorem, evaluate $\int_C (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$, where C is the boundary of the Triangle with vertices $(2,0,0)$, $(0,3,0)$, $(0,0,6)$.

Sol: Stoke's theorem states that $\int_C \bar{F}\cdot d\bar{r} = \int_S \operatorname{Curl}\bar{F}\cdot\bar{n}\,ds$

$$\text{Curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\bar{i} + \bar{k}$$

Equation of the plane through the points $(2,0,0), (0,3,0), (0,0,6)$ is $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1 \Rightarrow 3x+2y+z=6$

Normal to this plane is $\nabla(3x+2y+z-6) = 3\bar{i}+2\bar{j}+\bar{k}$

Unit normal $\bar{n} = \frac{1}{\sqrt{14}}(3\bar{i}+2\bar{j}+\bar{k})$

Now $\int_C (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$

$$= \int_S \text{Curl } \bar{F} \cdot \bar{n}\, ds$$

$$= \int_S \frac{1}{\sqrt{14}}(6+1)\, ds$$

$$= \frac{7}{\sqrt{14}} \int_S ds$$

$$= \frac{7}{\sqrt{14}}(\text{Area of } \Delta ABC)$$

$$= \frac{7}{\sqrt{14}}(3\sqrt{14})$$

$$= 21$$

$$\text{Area} = \frac{1}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}$$

$$= \frac{1}{2}\sqrt{36+324+144}$$

$$= \frac{1}{2}\sqrt{504}$$

$$= 3\sqrt{14}$$

Prepared by :
B. Bharathi
Dr. B. Bharathi
Assistant Professor
Dept. of BS&H
GVPCEW (A).